J. Fluid Mech. (1982), *vol.* 115, *pp.* 315–325
Printed in Great Britain

On surface-wave diffraction by a trench

By JOHN W. MILES
Institute of Geophysics and Planetary Physics,
University of California, San Diego

(Received 5 February 1981 and in revised form 14 July 1981)

The two-dimensional diffraction of a long surface wave by a deformation of the bottom is calculated through a conformal-mapping algorithm (Kreisel 1949). The result is applied to obtain the complex reflection coefficient for a rectangular trench. The corresponding reflection coefficient for oblique incidence is obtained through a variational formulation.

1. Introduction

The two-dimensional diffraction of surface waves by a submarine trench has been treated by Lassiter (1972) and Lee & Ayer (1981), but they do not give results that are directly useful for long waves (e.g. tsunamis), which are likely to be of the greatest oceanographic interest.

The long-wave problem for any bottom deformation of dimensions small compared with the wavelength can be solved by conformal mapping and is closely related to the problem of determining the incremental capacitance associated with the corresponding deformation of a two-dimensional, parallel-plate capacitor. This approach to diffraction problems goes back at least to Rayleigh (1897) and Lamb (1898), and has been further developed by Schwinger (*c.* 1944 at the M.I.T. Radiation Laboratory; see Schwinger & Saxon (1968)), Kreisel (1949) and Tuck (1976). Kreisel's result for the magnitude of the reflection coefficient of a small obstacle provides perhaps the most elegant basis for the present calculation; however, he omits the derivation and does not give the phase. (Tuck (1976) obtains both the amplitude and the phase of the reflection coefficient, but his results are in less compact form than that of Kreisel.) I sketch the derivation of Kreisel's result in § 2 and obtain explicit results for a rectangular trench in § 3.

The reflection coefficient for a wave that is obliquely incident upon a trench may be obtained from a slight generalization of Lassiter's (1972) variational formulation, following Miles (1967); however, since Lassiter's formulation is for the asymmetric problem (with different depths on the two sides of the trench), it proves simpler to alter Mei & Black's (1969) variational formulation for diffraction by a rectangular obstacle. I do this in § 4 and then take the long-wave limit to obtain a rather simple correction for the effect of obliquity on the reflection coefficient.

2. Long-wave formulation

The gravity wave

$$y = \eta(x,t) = \mathscr{R}\{A\, e^{i(\kappa x-\sigma t)}\}, \tag{2.1}$$

where κ is determined by

$$\kappa \tanh \kappa h = \sigma^2/g \equiv \kappa_\infty, \tag{2.2}$$

is incident, from $x = -\infty$, on a two-dimensional obstacle, $y = -h+f(x)$ ($|f| < h$), of finite breadth on the bottom of a laterally unbounded, inviscid ocean of ambient depth h. The corresponding velocity potential may be posed in the form

$$\phi(x,y,t) = \mathscr{R}\{\Phi(x,y)\, e^{-i\sigma t}\}, \tag{2.3}$$

where Φ satisfies

$$\Phi_{xx} + \Phi_{yy} = 0 \quad (-h+f < y < 0), \tag{2.4}$$

$$\Phi_y = \kappa_\infty \Phi, \tag{2.5}$$

$$\Phi_y = f'\Phi_x \quad (y = -h+f), \tag{2.6}$$

and has the asymptotic forms

$$\Phi \sim -i\sigma A(\kappa \sinh \kappa h)^{-1} \begin{Bmatrix} T\,e^{i\kappa x} \\ e^{i\kappa x} + Re^{-i\kappa x} \end{Bmatrix} \cosh \kappa(y+h) \quad (x \to \pm\infty), \tag{2.7}$$

in which R and T are the reflection and transmission coefficients for the obstacle and, here and subsequently, the alternatives are vertically ordered.

We now assume that both the depth h and the lateral dimensions of the obstacle are small compared with g/σ^2, so that (2.2), (2.5) and (2.7) may be replaced by

$$\kappa h = (\kappa_\infty h)^{\frac{1}{2}} \ll 1, \tag{2.8}$$

$$\Phi_y = 0 \quad (y = 0), \tag{2.9}$$

$$\Phi \sim (gA/i\sigma)\left\{\begin{matrix} T \\ 1+R \end{matrix} + i\kappa x \begin{pmatrix} T \\ 1-R \end{pmatrix}\right\} \quad (h \ll \pm x \ll 1/\kappa). \tag{2.10}$$

The reduced boundary-value problem described by (2.4), (2.6), (2.9) and (2.10) corresponds to potential flow in the two-dimensional channel $-h+f < y < 0$ with the asymptotic velocities

$$U_\pm = \frac{\kappa g A}{\sigma}\begin{pmatrix} T \\ 1-R \end{pmatrix} \equiv U, \tag{2.11}$$

which must be equal by virtue of continuity; accordingly, we may rewrite (2.10) in the form

$$\Phi \sim U(x \pm l) + (i\sigma)^{-1} gA \quad (b+h \ll \pm x \ll 1/\kappa), \tag{2.12}$$

where the length l is determined by comparing (2.12) with (2.10) and invoking (2.11). It follows from this comparison that

$$R = -i\kappa l(1-i\kappa l)^{-1}, \quad T = (1-i\kappa l)^{-1}. \tag{2.13a, b}$$

We remark that the approximations (2.13a, b) satisfy the conservation-of-energy requirements $|R|^2+|T|^2 = 1$ and $|R+T| = 1$ exactly and are valid through $O(\kappa^2 l^2)$, whereas the cruder approximations $R = -i\kappa l$ and $T = 1+i\kappa l$ are in error by $O(\kappa^2 l^2)$.

The solution of the potential-flow problem is obtained by mapping $-h+f(x) < y < 0$

in the $(x+iy)$-plane on the strip $-h < \mathscr{I}(\zeta) < 0$ in the complex ζ-plane, where $\Phi = U\mathscr{R}(\zeta)$ and the mapping is scaled so that $dz/d\zeta \sim 1$ as $\zeta \to \infty$. It then follows from (2.11) that (cf. Kreisel 1949)

$$2l = \int_{-\infty}^{\infty} \left(1 - \frac{dz}{d\zeta}\right)_{y=0} d\zeta, \tag{2.14}$$

where $z \equiv x + iy$.

The dimensionless parameter $2l/h$ may be identified as the incremental electrostatic capacitance per unit width associated with the obstacle, qua deformation of the parallel-plate capacitor formed by perfect conductors at $y = 0, -h$. It follows that various isoperimetric inequalities are available for its estimation (Pólya & Szegö 1951); in particular, upper and lower bounds to l may be obtained by replacing the profile of the obstacle by curves that bound it from above or below.

It may be inferred from (2.13*a*) and Kreisel's (1949) result for a low, gently sloping obstacle that (2.14) has the limiting form

$$2l \sim S/h \quad (|f(x)|/h,\ |f'(x)| \downarrow 0), \tag{2.15}$$

where S is the cross-sectional area of the obstacle and is negative for a depression.

3. Rectangular trench

We now calculate l for a rectangular trench of breadth $2b$ and depth d (figure 1*a*). The interior of the polygon $ABCDEFGH$ is mapped on the half-plane $\mathscr{I}(w) > 0$ (figure 1*b*) by

$$\frac{dz}{dw} = \frac{2h}{\pi}\left(\frac{1-a^2}{k^2-a^2}\right)^{\frac{1}{2}}\left(\frac{1-k^2w^2}{1-w^2}\right)^{\frac{1}{2}}\left(\frac{a}{1-a^2w^2}\right) \quad (0 < a < k < 1), \tag{3.1}$$

which, in turn, is mapped on the strip $-h < \mathscr{I}(\zeta) < 0$ (figure 1*c*), with $z \sim \zeta$ as $x \to \pm\infty$, by

$$\frac{d\zeta}{dw} = \frac{2h}{\pi}\left(\frac{a}{1-a^2w^2}\right). \tag{3.2}$$

The transformations are scaled so that both z and ζ increase by ih as the pole at $w = 1/a$ is traversed from left to right. The dimensionless parameters a and k are implicitly determined by b/h and d/h; however, it is expedient to regard a and k as the primary family parameters, as functions of which b/h, d/h and l/h are to be determined.

These determinations are expedited by the further transformation

$$w = \operatorname{sn}(\tau, k), \tag{3.3}$$

which maps $\mathscr{I}(w) > 0$ on the interior of a rectangle with vertices at $\tau = \pm K$ and $\pm K + iK'$ (figure 1*d*); sn is a Jacobi elliptic function of modulus k, K is a complete elliptic integral of the first kind and modulus k, and K' is the complementary integral of modulus $k' \equiv (1-k^2)^{\frac{1}{2}}$. The notation follows Byrd & Friedman (1954), hereinafter referenced by the prefix BF, followed by the number of the appropriate entry therein. It also is convenient to introduce

$$\beta = \sin^{-1}(a/k), \quad T = \operatorname{sn}^{-1}(a/k, k) \equiv F(\beta, k), \tag{3.4a, b}$$

where $F(\beta, k)$ is an incomplete elliptic integral of the first kind, and

$$C = a\left(\frac{1-a^2}{k^2-a^2}\right)^{\frac{1}{2}} = (1 - k^2\sin^2\beta)^{\frac{1}{2}}\tan\beta = \frac{\operatorname{sn} T \operatorname{dn} T}{\operatorname{cn} T}, \tag{3.5}$$

where $\operatorname{sn} T \equiv \operatorname{sn}(T, k)$, $\operatorname{cn} T$ and $\operatorname{dn} T$ are Jacobi elliptic functions.

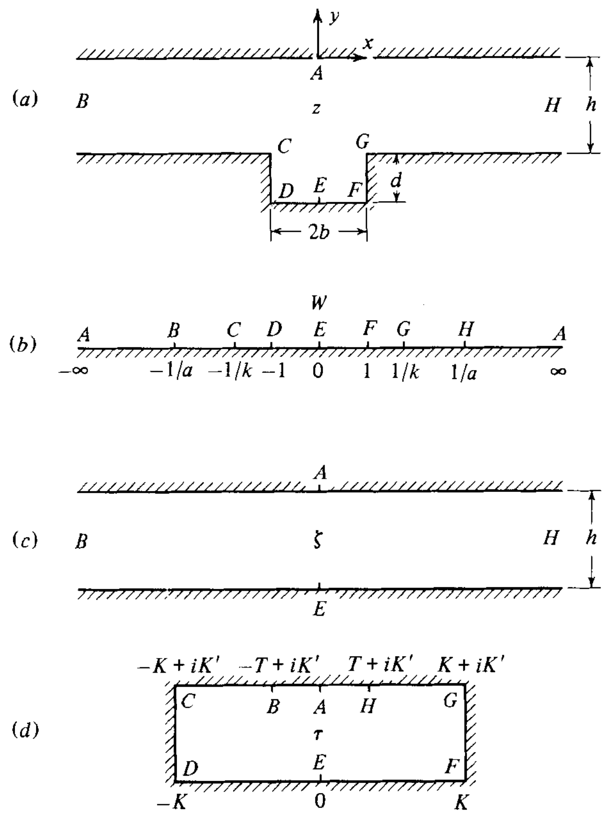


FIGURE 1. (*a*) Rectangular trench in $z = x+iy$ plane. (*b*) Mapping on w-plane, as given by (3.1). (*c*) Mapping on ζ-plane, as given by (3.2). (*d*) Mapping on τ-plane, as given by (3.3).

The parameters b/h and d/h, as determined by integrating (3.1) between $w = 0$ and $1/k$ and invoking (3.3) and BF 435.04†, are given by (cf. Carter 1926)

$$\frac{b+id}{h} = \frac{2C}{\pi}\int_0^{K+iK'} \frac{\mathrm{dn}^2\tau\, d\tau}{1-a^2\,\mathrm{sn}^2\tau} \tag{3.6a}$$

$$= \frac{2}{\pi}(K+iK')\{C-Z(\beta,k)\} - i\frac{T}{K}, \tag{3.6b}$$

where

$$Z(\beta,k) = E(\beta,k)-(E/K)\,F(\beta,k) \tag{3.7}$$

is Jacobi's Zeta function (BF 140), $E(\beta,k)$ and $F(\beta,k)$ are incomplete elliptic integrals of the second and first kinds, respectively, and E is a complete elliptic integral of the second kind and modulus k.

The parameter l/h, as determined by substituting (3.1)–(3.3) into (2.14), letting $\tau = t+iK'$, and invoking BF 122.07, is given by

$$\frac{l}{h} = -\frac{2}{\pi}\,\mathrm{tn}\,T\int_0^T \left(\frac{\mathrm{dn}\,T\,\mathrm{cn}\,t-\mathrm{cn}\,T\,\mathrm{dn}\,t}{\mathrm{sn}^2 T-\mathrm{sn}^2 t}\right)\mathrm{cn}\,t\,dt. \tag{3.8}$$

† The Fourier series Ω_3 in BF 435.04 has been summed to obtain the last term in (3.6*b*).

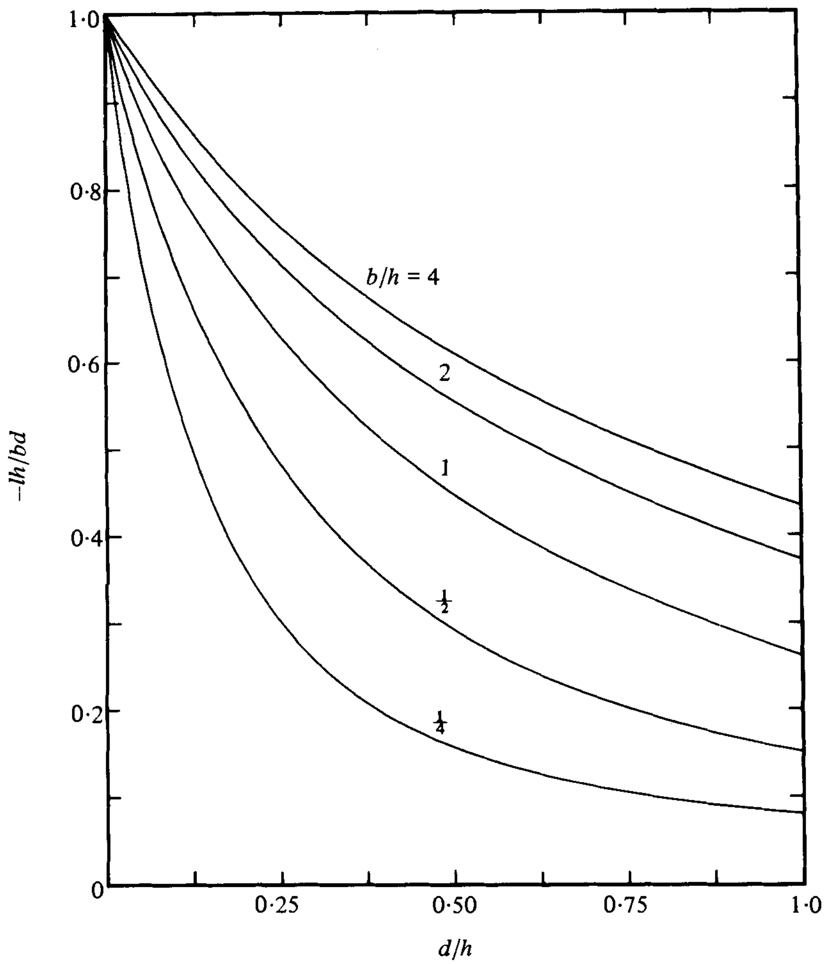


FIGURE 2. The dimensionless parameter $-lh/bd$, as calculated from (3.6) and (3.9) for $b/h = \frac{1}{4}, \frac{1}{2}, 1, 2, 4$.

The integrand in (3.8) is indeterminate, but has the limiting value $\frac{1}{2}k'^2 \,\mathrm{nd}\, T$, at $t = T$. Replacing T by t_1, invoking BF 436.03, and letting $t_1 \uparrow T$, we obtain

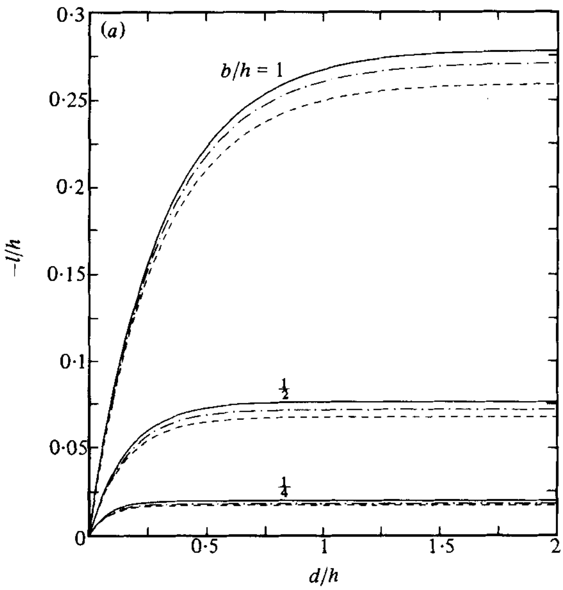
$$\frac{l}{h} = -\frac{T}{K}\frac{b}{h} + \frac{1}{\pi}\ln\left\{\frac{\pi \,\mathrm{tn}\, T \,\mathrm{nd}\, T}{K\sin(\pi T/K)}\right\} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\exp(-n\pi K'/K)\sin^2(n\pi T/K)}{n\sinh(n\pi K'/K)}. \quad (3.9)$$

Numerical results calculated from (3.6) and (3.9) are plotted in figures 2 and 3.†

Letting $k \uparrow 1$ $(d/h \downarrow 0)$ in (3.6) and (3.8), we obtain

$$\frac{b}{h} \sim \frac{2}{\pi}\tanh^{-1} a, \quad \frac{d}{h} \sim \left(\frac{1-k^2}{2}\right)\left(\frac{a}{1-a^2}\right), \quad \frac{l}{h} \sim -\frac{bd}{h^2} \quad (k \uparrow 1). \quad (3.10a, b, c)$$

† J. N. Newman (private communication) has pointed out that my result for $1-(lh/bd)$ should be equivalent to the added-mass coefficient $\lambda_x/4\rho bd$ of Flagg & Newman (1971) for a rectangular bump, continued for negative values of d. Graphical interpolation between the two results suggests that this is likely to be so, but I have been unable to demonstrate analytical equivalence.



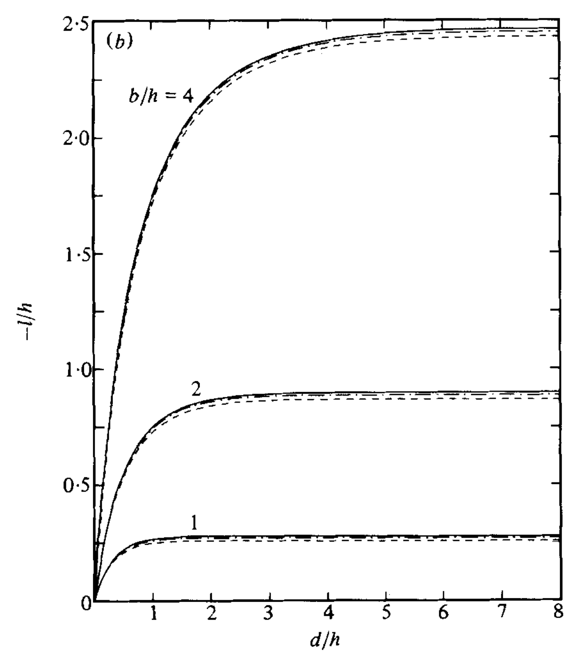


FIGURE 3. The dimensionless parameter $-l/h$, as calculated from (3.6) and (3.9) (——), (4.26a) and (4.30a) (– – –), and (4.26a) and (4.30b) (– · –), for (a) $b/h = \frac{1}{4}, \frac{1}{2}, 1$; ($b$) $b/h = 1, 2, 4$.

We remark that (3.10c) agrees with the limiting approximation (2.15) even though $f'(x)$ is infinite at the ends of the trench; however, the approximation is satisfactory only for quite small d/h.

Letting $k \downarrow 0$ $(d/h \uparrow \infty)$ with β fixed in (3.6) and (3.8), we obtain

$$\frac{b}{h} \sim \tan\beta, \quad \frac{d}{h} \sim \frac{2}{\pi}\left(\tan\beta \ln\frac{4}{k} - \beta\right), \quad \frac{l}{h} \sim -\frac{2}{\pi}(\beta\tan\beta - \ln\sec\beta) \quad (k \downarrow 0), \tag{3.11 a, b, c}$$

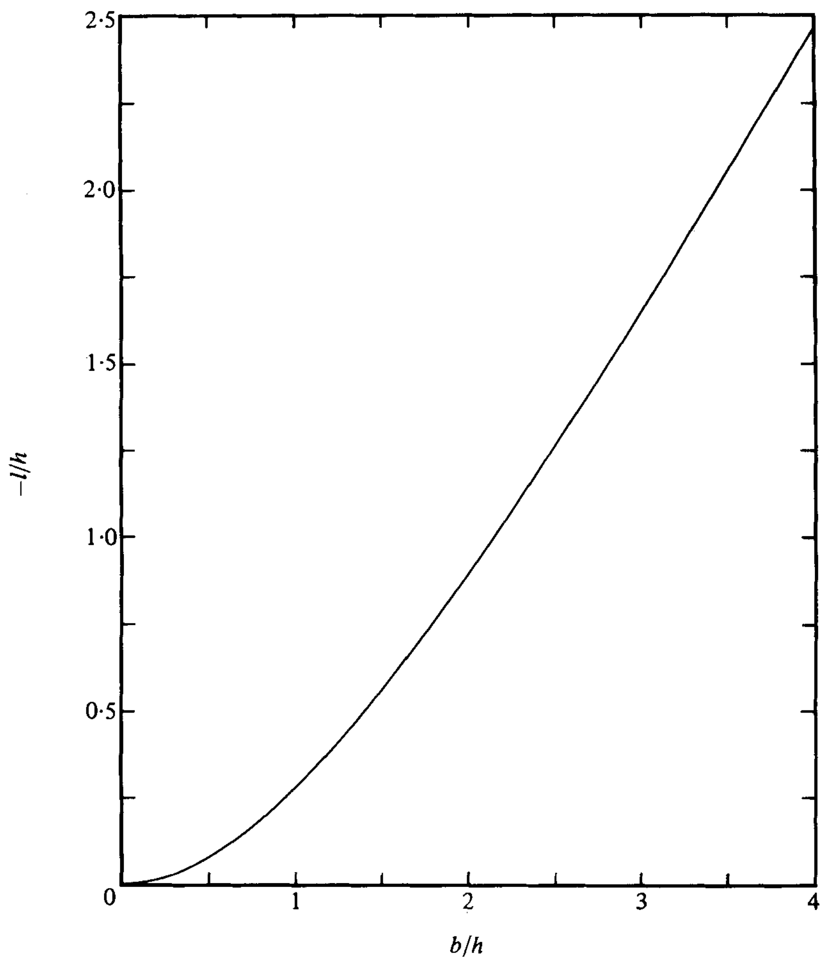


FIGURE 4. The dimensionless parameter $-l/h$ for $d/h \gg 1$, as calculated from (3.12).

or, equivalently,

$$l \sim -\frac{2b}{\pi}\tan^{-1}\frac{b}{h}+\frac{h}{\pi}\ln\left(1+\frac{b^2}{h^2}\right) \quad \left(\frac{d}{h}\uparrow\infty\right), \tag{3.12}$$

which is plotted in figure 4.

4. Oblique incidence; variational formulation

Waves moving obliquely with respect to the trench of figure 1(*a*) are described by

$$\phi(x,y,z,t) = \mathscr{R}\{\Phi(x,y)\, e^{i(\gamma z-\sigma t)}\}, \tag{4.1}$$

where the z-axis is directed along the trench (the complex variable z occurs only in § 3), γ is the z-component of the wavenumber (we assume $\gamma < \kappa$), Φ satisfies

$$\Phi_{xx}+\Phi_{yy}-\gamma^2\Phi = 0, \tag{4.2}$$

$$\Phi_y = \kappa_\infty \Phi \quad (y = 0), \tag{4.3}$$

$$\Phi_x = 0 \quad (x = \pm b,\ -H < y < -h), \tag{4.4a}$$

$$\Phi_y = 0 \quad (|x| > b, y = -h; |x| < b, y = -H), \tag{4.4b}$$

and $H \equiv h+d$ is the total depth of the trench.

Geometric symmetry permits the scattering problem for the trench to be separated into symmetric and antisymmetric problems. We pose the solution for the symmetric problem (the basic symmetry is that of the velocity potential, which is the same as that of the free-surface displacement but opposite that of the velocity ϕ_x) in the form

$$\Phi_s = \{A_s e^{-i\alpha(|x|-b)} + B_s e^{i\alpha(|x|-b)}\}\chi(y) - \sum_k{}' (k^2+\gamma^2)^{-\frac{1}{2}} \exp\{-(k^2+\gamma^2)^{\frac{1}{2}}(|x|-b)\}\psi_k(y)\int_{-h}^{0} U_s(\eta)\psi_k(\eta)\,d\eta \quad (|x| > b), \tag{4.5$_s$}$$

$$\Phi_s = \sum_K \frac{\cosh\{(K^2+\gamma^2)^{\frac{1}{2}}x\}\Psi_K(y)}{(K^2+\gamma^2)^{\frac{1}{2}}\sinh\{(K^2+\gamma^2)^{\frac{1}{2}}b\}}\int_{-h}^{0} U_s(\eta)\Psi_K(\eta)\,d\eta \quad (|x| < b), \tag{4.6$_s$}$$

where A_s is the amplitude of the incoming (towards the trench) waves, B_s is the amplitude of the outgoing waves,

$$\alpha = (\kappa^2 - \gamma^2)^{\frac{1}{2}} \quad (\gamma < \kappa) \tag{4.7}$$

is the x-component of the wavenumber,

$$\psi_k = (2/h)^{\frac{1}{2}}\{1+(2kh)^{-1}\sin 2kh\}^{-\frac{1}{2}}\cos k(h+y), \quad k\tan kh = -\kappa_\infty, \tag{4.8a, b}$$

and

$$\Psi_K = (2/H)^{\frac{1}{2}}\{1+(2KH)^{-1}\sin 2KH\}^{-\frac{1}{2}}\cos K(H+y), \quad K\tan KH = -\kappa_\infty. \tag{4.9a, b}$$

The ψ_k and Ψ_K are complete, orthonormal sets of functions in $-h < y < 0$ and $-H < y < 0$, respectively, for which the eigenvalues are determined by (4.8*b*) and (4.9*b*) (K and k stand for roots of (4.9*b*) and (4.8*b*), respectively, throughout this section, rather than for an elliptic integral and its modulus as in §3). The primed summation in (4.5)$_s$ is over the positive real roots of (4.8*b*) but excludes the imaginary root $k = i\kappa$, for which (4.8*a*) has the equivalent form

$$\psi_{i\kappa} \equiv \chi = (2/h)^{\frac{1}{2}}\{1+(2\kappa h)^{-1}\sinh 2\kappa h\}^{-\frac{1}{2}}\cosh\{\kappa(h+y)\}. \tag{4.10}$$

The summation in (4.6)$_s$ is over the real roots of (4.9*b*) and the positive imaginary root if such a root exists (there is a finite range of γ for which no such root exists, in which case the obliquely incident wave would be totally reflected at an increase of depth from h to H). The coefficients of ψ_k and Ψ_K in these summations have been determined from the corresponding Fourier expansions of U_s, the former of which also implies

$$-i\alpha(A_s - B_s) = \int_{-h}^{0} U_s(y)\chi(y)\,dy. \tag{4.11$_s$}$$

The continuity of Φ across $x = b$ implies the integral equation

$$\int_{-h}^{0} G_s(y,\eta)U_s(\eta)\,d\eta = (A_s+B_s)\chi(y) \quad (-h < y < 0), \tag{4.12$_s$}$$

where

$$G_s = \sum_k{}'(k^2+\gamma^2)^{-\frac{1}{2}}\psi_k(y)\psi_k(\eta) + \sum_K (K^2+\gamma^2)^{-\frac{1}{2}}\coth\{(K^2+\gamma^2)^{\frac{1}{2}}b\}\Psi_K(y)\Psi_K(\eta). \tag{4.13$_s$}$$

It is expedient, at this point, to normalize U_s and introduce the length L_s† according to

$$U_s = -i\alpha(A_s - B_s)\, u_s, \quad A_s + B_s = -i\alpha(A_s - B_s)\, L_s, \tag{4.14a, b)$_s$}$$

the substitution of which into $(4.11)_s$ and $(4.12)_s$ yields

$$\int_{-h}^{0} u_s(y)\, \chi(y)\, dy = 1, \qquad (4.15)_s$$

$$\int_{-h}^{0} G_s(y, \eta)\, u_s(\eta)\, d\eta = L_s \chi(y). \qquad (4.16)_s$$

Multiplying $(4.16)_s$ through by $u_s(y)$, integrating over $-h < y < 0$, and invoking $(4.15)_s$,we obtain the variational integral

$$L_s = \int_{-h}^{0}\int_{-h}^{0} u_s(y)\, G_s(y, \eta)\, u_s(\eta)\, d\eta\, dy, \qquad (4.17)_s$$

which is an absolute minimum with respect to variations of $u_s(y)$ about the solution to the integral equation $(4.16)_s$, subject to the constraint $(4.15)_s$. The constraint may be eliminated by dividing $(4.17)_s$ through by the square of $(4.15)_s$ to obtain

$$L_s = \left\{\int_{-h}^{0} u_s(y)\, \chi(y)\, dy\right\}^{-2} \int_{-h}^{0}\int_{-h}^{0} u_s(y)\, G_s(y, \eta)\, u_s(\eta)\, d\eta\, dy, \qquad (4.18)_s$$

which is invariant under a scale transformation of u_s.

The formulation of the antisymmetric problem parallels $(4.5)_s$–$(4.18)_s$, with the following changes: the subscript s is replaced by a; the sign of the right-hand side of $(4.5)_s$ is reversed in $x < -b$; cosh and sinh in the numerator and denominator of $(4.6)_s$ are replaced by sinh and cosh, respectively; coth in $(4.13)_s$ is replaced by tanh.

It follows from $(4.5)_s$ and $(4.5)_a$ that the sum and difference of the reflection and transmission coefficients (note that each of the two incoming waves in the symmetric problem produces both a reflected and a transmitted wave and similarly for the antisymmetric problem) are given by

$$R + T = (B_s/A_s) \exp(-2i\alpha b) \equiv \exp(2i\theta_s), \qquad (4.19)_s$$

$$R - T = (B_a/A_a) \exp(-2i\alpha b) \equiv \exp(2i\theta_a), \qquad (4.19)_a$$

where

$$\theta_s = \tfrac{1}{2}\pi - \alpha b + \tan^{-1} \alpha L_s \qquad (4.20)_s$$

follows from $(4.14b)_s$ and similarly for θ_a. It follows from $(4.19)_s$ and $(4.19)_a$ that

$$R = \tfrac{1}{2}\{\exp(2i\theta_s) + \exp(2i\theta_a)\}, \tag{4.21a}$$

$$T = \tfrac{1}{2}\{\exp(2i\theta_s) - \exp(2i\theta_a)\}. \tag{4.21b}$$

We turn now to the long-wave limit. Letting $\kappa_\infty h \downarrow 0$ with H/h and $b/h = O(1)$ in (4.8)–(4.10) and substituting the resulting approximations,

$$k = i(\kappa_\infty/h)^{\frac{1}{2}}, \quad n\pi/h \quad (n = 1, 2, \ldots), \quad \psi_k = h^{-\frac{1}{2}}, \quad (-)^n (2/h)^{\frac{1}{2}} \cos(n\pi y/h), \tag{4.22}$$

† The inverse length $1/L_s$ is a degenerate form of the scattering matrix in the corresponding formulation of asymmetric scattering problems (Miles 1967).

and similarly for K and Ψ_K, into $(4.13)_{s,a}$ and $(4.18)_{s,a}$, we obtain

$$\frac{L_s}{h} = -\frac{1}{(\kappa_\infty - \gamma^2 H)\,b} + O(1), \quad \frac{L_a}{h} = \frac{b}{H} + \Lambda + O(\kappa_\infty h), \tag{4.23a, b}$$

where

$$\Lambda = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\left\{\int_{-h}^{0} u_a \cos(n\pi y/h)\,dy\right\}^2 + \tanh(n\pi b/H)\left\{\int_{-h}^{0} u_a\cos(n\pi y/H)\,dy\right\}^2}{\left(\int_{-h}^{0} u_a\,dy\right)^2}\right]. \tag{4.24}$$

Substituting (4.23a, b) into $(4.20)_{s,a}$, invoking (2.8), (4.7) and (4.21a, b), and retaining only the dominant (as $\kappa_\infty h \downarrow 0$) terms, we obtain

$$\theta_s = \pi + \alpha l_s + O(\kappa_\infty h), \quad \theta_a = \tfrac{1}{2} + \alpha l_a + O(\kappa_\infty h), \tag{4.25a, b}$$

where

$$l_s = -(\gamma/\alpha)^2 (bd/h), \quad l_a = -bdH^{-1} + h\Lambda, \tag{4.26a, b}$$

$$R = \tfrac{1}{2}(e^{2i\alpha l_s} - e^{2i\alpha l_a}) = -i\,e^{i\alpha(l_s+l_a)}\sin\alpha(l_a - l_s), \tag{4.27a}$$

$$T = \tfrac{1}{2}(e^{2i\alpha l_s} + e^{2i\alpha l_a}) = e^{i\alpha(l_s+l_a)}\cos\alpha(l_a - l_s). \tag{4.27b}$$

Error factors of $1 + O(\kappa_\infty h)$ are implicit in the exponents in (4.27a, b). The corresponding first-order approximations are

$$R = -i\alpha(l_a - l_s), \quad T = 1 + i\alpha(l_a + l_s). \tag{4.28a, b}$$

Note, however, that (4.27) imply $|R|^2 + |T|^2 = 1$ exactly, whereas (4.28) do so only approximately.

It is evident from (4.24) and the fact that u_a may be expanded in the complete set of functions $\cos(n\pi y/h)$ $(n = 0, 1, 2, \ldots)$ that Λ is independent of γ. It follows that l_a, as given by (4.26b), also is independent of γ and therefore must be equivalent to l, as defined in § 2 and calculated in § 3; accordingly, (4.27) provides the extension of (2.13) to oblique incidence through $O(\kappa^2 l^2)$. We remark that $R = 0$ at those angles of incidence for which $(\gamma/\alpha)^2 = -hl/bd$ and that these angles are $\pm 45°$ for a very shallow trench $(d \ll h)$.

Returning to (4.24), we consider the variational approximations implied by the truncated Fourier expansions

$$u_0 = 1, \quad u_1 = 1 + C\cos(\pi y/h) \tag{4.29a, b}$$

(recall that Λ is invariant under a scale transformation of u_a, so that the dominant term in the Fourier expansion may be taken as unity). Substituting (4.29a, b) into (4.24) and minimizing Λ to determine C, we obtain

$$\Lambda_0 = S_0, \quad \Lambda_1 = S_0 - 2\pi S_1^2 (1 + 2\pi S_2)^{-1}, \tag{4.30a, b}$$

respectively, where

$$S_m = 2\left(\frac{H}{h}\right)^2 \sum_{n=1}^{\infty}\left\{1 - \left(\frac{H}{nh}\right)^2\right\}^{-m} \frac{\tanh(n\pi b/H)\sin^2(n\pi h/H)}{(n\pi)^3} \quad (m = 0, 1, 2). \tag{4.31}$$

Note that $\Lambda_1 < \Lambda_0$, in accordance with the variational principle. The corresponding approximations to l/h are plotted in figure 3. It is evident that the variational approximations are quite accurate and offer the advantage of displaying the explicit dependence of l/h on b/h and d/h.

This work was partially supported by the Physical Oceanography Division, National Science Foundation, NSF Grant OCE 77-24005, and by a contract with the Office of Naval Research.

REFERENCES

BYRD, P. F. & FRIEDMAN, M. D. 1954 *Handbook of Elliptic Integrals for Engineers and Physicists*. Springer.

CARTER, F. W. 1926 The electric field of the dynamo-electric machine. *Proc. I. E. E.* **64**, 1115–1138.

FLAGG, C. N. & NEWMAN, J. N. 1971 Sway added-mass coefficients for rectangular profiles in shallow water. *J. Ship Res.* **15**, 257–265.

KREISEL, G. 1949 Surface waves. *Quart. Appl. Math.* **7**, 21–44.

LAMB, H. 1898 On the diffraction in transmission of electric waves by a metallic grating. *Proc. Lond. Math. Soc.* **29**, 523–544.

LASSITER, J. B. 1972 The propagation of water waves over sediment pockets. Doctoral dissertation, M.I.T.

LEE, J. J. & AYER, R. M. 1981 Wave propagation over a rectangular trench. *J. Fluid Mech.* **110**, 335–347.

MEI, C. C. & BLACK, J. 1969 Scattering of surface waves by rectangular obstacles in waters of finite depth. *J. Fluid Mech.* **38**, 499–511.

MILES, J. W. 1967 Surface-wave scattering matrix for a shelf. *J. Fluid Mech.* **28**, 755–767.

PÓLYA, G. & SZEGÖ, G. 1951 *Isoperimetric Inequalities in Mathematical Physics*. Princeton University Press.

RAYLEIGH, LORD 1897 On the passage of waves through apertures in plane screens, and allied problems. *Phil. Mag.* **43**, 259–272 (*Scientific Papers*, vol. **4**, pp. 283–296. Cambridge University Press).

SCHWINGER, J. & SAXON, D. S. 1968 *Discontinuities in Waveguides*. Gordon & Breach.

TUCK, E. O. 1976 Some classical water-wave problems in varying depth. In *Waves on Water of Varying Depth* (ed. D. G. Provis & R. Radok). Springer.